MEI Structured Mathematics

Practice Comprehension Question - 1

(Concepts for Advanced Mathematics, C2)

Pythagoran and other Triples

Pythagoras was born on the Greek island of Samos in BC572 and founded a school of philosophy, mathematics and natural science in the Greek seaport of Crotona in Southern Italy. He is, of course, famous for the theorem that bears his name, that in a right angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.

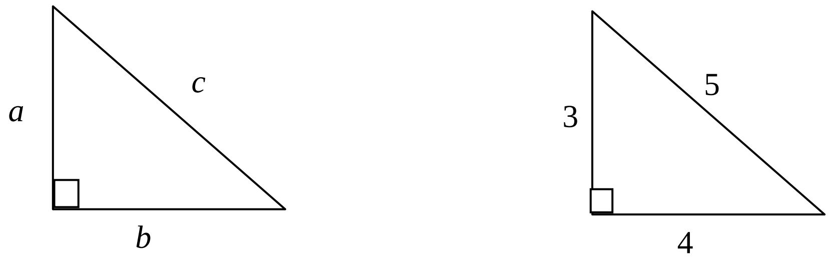

Fig. 1

$a^2 + b^2 = c^2$ $\qquad\qquad$ $3^2 + 4^2 = 5^2$ $(9 + 16 = 25)$

A set of three integers which can form the sides of a right-angled triangle, such as {3,4,5} is called a Pythagoran Triple.

One way of generating Pythagoran Triples is as follows:

	Example	**Algebraically**
Think of an odd number	7	$2n + 1$
Square it	49	$4n^2 + 4n + 1$
Split your answer into two numbers, one smaller by one than the other	24,25	$2n^2 + 2n,\ 2n^2 + 2n + 1$
The number you first thought of and these two form a Pythagoran Triple.	{7, 24, 25}	$\{2n + 1, 2n^2 + 2n,\ 2n^2 + 2n + 1\}$

There are other Pythagoran Triples, such as {8, 15, 17} which cannot be generated by this method.
It can, however, be shown that all Pythagoran Triples are generated by the formula

$$\{2mn,\ m^2 - n^2,\ m^2 + n^2\}$$

Where m and n are integers. For instance, substituting $m = 4$ and $n = 1$ gives the Triple {8, 15, 17}.

There are also triangles with one angle 60^0 and sides of integer length, like the two shown in Fig. 2 below and a generator for these triangles is

$$\{p^2 + 2pq,\ 2pq + q^2,\ p^2 + pq + q^2\}$$

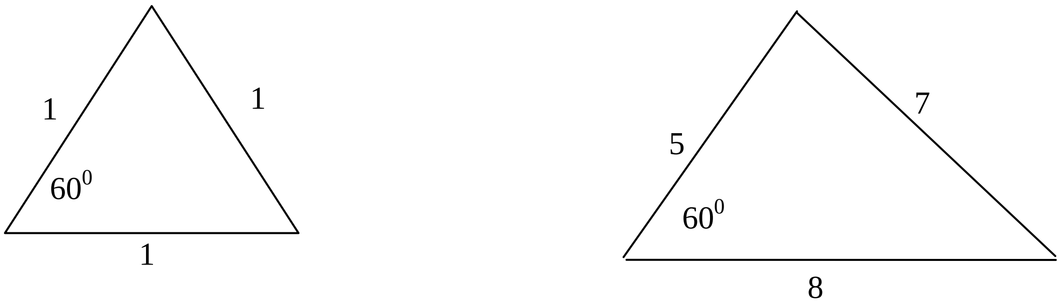

Fig. 2

Questions:

1 Pythagoras Theorem states:
In a right-angled triangle the sum of the squares of two sides is equal to the square on the hypotenuse.
State the converse of this theorem. [1]

2 In Table 1, the number 7 is used as a starting point in the example. Follow the procedure starting with the number 11 to produce a Pythagorean Triple. [2]

3 Explain why a general odd number has the form $2n + 1$. [1]

4 Prove that $\{2n + 1, 2n^2 + 2n, \ 2n^2 + 2n + 1\}$ is a pythagoran triangle. [2]

5 In line 12 the generator for Pythagoran Triples is given.

(i) Show that this Triple is a Pythagoran Triangle for all positive integers m, n where you may take $m \geq n$. [2]

(ii) Find the Triple generated by $m = 3$ and $n = 2$. [1]

(iii) Find the values of m and n which generate the triple {140, 51, 149}. [2]

(iv) Copy and continue the table below to generate 5 *different* triples.
(In this context {6, 8, 10} is a multiple of [3, 4, 5] and so is not different.) [2]

m	n	$2mn$	$m^2 - n^2$	$m^2 + n^2$
2	1	4	3	5
3	1	6	8	10
3	2			

6 **(i)** In Fig. 2 there is an example given with sides 5, 8 and 7 where an angle is 60^0. Find the values of p and q that generate this Triple. [2]

(ii) Prove that with the generator given in line 16, one angle is indeed 60^0. [3]

Answers.

<table>
<tr><td>1</td><td></td><td>In a triangle, if the square of the longest side is equal to the sum of squares on the two shorter sides then the triangle is right-angled.</td><td>B1
1</td><td></td></tr>
<tr><td>2</td><td></td><td>11
121
65,66
giving {11, 65, 66}</td><td>M1
A1
2</td><td></td></tr>
<tr><td>3</td><td></td><td>$2n$ must be even for all n because it has a factor of 2.
therefore $2n + 1$ must be odd.</td><td>B1
1</td><td></td></tr>
<tr><td>4</td><td></td><td>$(2n+1)^2+(2n^2+2n)^2$
$=4n^2+4n+1+4n^4+8n^3+4n^2$
$=4n^4+8n^3+8n^2+4n+1$
$(2n^2+2n+1)^2=4n^4+4n^2(2n+1)+(2n+1)^2$
$=4n^4+8n^3+4n^2+4n^2+4n+1$
$=4n^4+8n^3+8n^2+4n+1$
So the sum of squares of two is equal to the square of the third.</td><td>M1
A1
2</td><td></td></tr>
<tr><td>5</td><td>(i)</td><td>$(2mn)^2+(m^2-n^2)^2=4m^2n^2+m^4-2m^2n^2+n^4$
$=m^4+2m^2n^2+n^4=(m^2+n^2)^2$</td><td>M1
A1
2</td><td></td></tr>
<tr><td></td><td>(ii)</td><td>12, 5, 13</td><td>B1
1</td><td></td></tr>
<tr><td></td><td>(iii)</td><td>We must have $2mn=140\Rightarrow mn=70$
$m^2-n^2=51$
$m^2+n^2=149$
Add: $2m^2=200\Rightarrow m^2=100$
$\Rightarrow m=10,\Rightarrow n=7$
i.e. {7, 51, 149}</td><td>M1
A1
2</td><td></td></tr>
<tr><td></td><td>(iv)</td><td>
<table>
<tr><td>m</td><td>n</td><td>2mn</td><td>m^2-n^2</td><td>m^2+n^2</td><td></td></tr>
<tr><td>2</td><td>1</td><td>4</td><td>3</td><td>5</td><td>No. 1</td></tr>
<tr><td>3</td><td>1</td><td>6</td><td>8</td><td>10</td><td>mult</td></tr>
<tr><td>3</td><td>2</td><td>12</td><td>5</td><td>13</td><td>No. 2</td></tr>
<tr><td>4</td><td>1</td><td>8</td><td>15</td><td>17</td><td>No. 3</td></tr>
<tr><td>4</td><td>2</td><td></td><td></td><td></td><td>Mult</td></tr>
<tr><td>4</td><td>3</td><td>24</td><td>7</td><td>25</td><td>No. 4</td></tr>
<tr><td>5</td><td>1</td><td>10</td><td>24</td><td>26</td><td>Mult.</td></tr>
<tr><td>5</td><td>2</td><td>20</td><td>21</td><td>29</td><td>No. 5</td></tr>
</table>
</td><td>M1
A1
2</td><td>some sort of methodical method
All 5</td></tr>
</table>

6	**(i)**	Either p or q must be equal to 1. This is because $p^2 + 2pq = p(p + 2q)$ and $2pq + q^2 = q(q + 2p)$ and one of them equals 5 which is prime. So take $p = 1$ giving $1 + 2q = 5 \Rightarrow q = 2$ $\{p^2 + 2pq, 2pq + q^2, p^2 + pq + q^2\}$	M1 A1 **2**	
	(ii)	By cosine rule: $\cos A = \dfrac{\left(p^2+2pq\right)^2+\left(q^2+2pq\right)^2-\left(p^2+pq+q^2\right)^2}{2\left(p^2+2pq\right)\left(q^2+2pq\right)}$ $= \dfrac{p^4+4p^3q+4p^2q^2+q^4+4pq^3+4p^2q^2-p^4-2p^2\left(pq+q^2\right)-\left(pq+q^2\right)^2}{2\left(p^2+2pq\right)\left(q^2+2pq\right)}$ $= \dfrac{2p^3q+2p^2q^2+q^4+4pq^3+4p^2q^2-p^2q^2-2pq^3-q^4}{2\left(p^2+2pq\right)\left(q^2+2pq\right)}$ $= \dfrac{2p^3q+5p^2q^2+2pq^3}{2\left(p^2+2pq\right)\left(q^2+2pq\right)} = \dfrac{\left(p^2+2pq\right)\left(q^2+2pq\right)}{2\left(p^2+2pq\right)\left(q^2+2pq\right)} = \dfrac{1}{2}$ $\Rightarrow A = 60^0$	M1 A1 A1 **3**	